ASSOCIATIVE CONES AND INTEGRABLE SYSTEMS

SHENGLI KONG, CHUU-LIAN TERNG*, AND ERXIAO WANG

Dedicated to the memory of Shing-Shin Chern

Abstract. We identify $\mathbb{R}^7$ as the pure imaginary part of octonions. Then the multiplication in octonions gives a natural almost complex structure for the unit sphere S^6. It is known that a cone over a surface M in S^6 is an associative submanifold of $\mathbb{R}^7$ if and only if M is almost complex in S^6. In this paper, we show that the Gauss-Codazzi equation for almost complex curves in S^6 is the equation for primitive maps associated to the 6-symmetric space G_2/T^2, and use this to explain some of the known results. Moreover, the equation for S^1-symmetric almost complex curves in S^6 is the periodic Toda lattice associated to G_2, and a discussion of periodic solutions is given.

1. Introduction

We identify $\mathbb{R}^7$ as the pure imaginary part of the octonions $\mathbb{O}$. It is known that the group of automorphism of $\mathbb{O}$ is the compact simple Lie group G_2, and the constant 3-form on $\mathbb{R}^7$,

$$\phi(u_1, u_2, u_3) = (u_1 \cdot u_2, u_3),$$

is invariant under G_2. A 3-dimensional submanifold M in $\mathbb{R}^7$ is *associative* if $\mathbb{R}1 + TM_x$ is an associative subalgebra of $\mathbb{O}$ for all $x \in M$, i.e., it is isomorphic to the quaternions. It is easy to see that a 3-dimensional submanifold of $\mathbb{R}^7$ is associative if and only if it is calibrated by the 3-form ϕ.

The multiplication of octonions defines an almost complex structure on the unit sphere S^6 by $J_x(v) = x \cdot v$. An immersion f from a Riemann surface Σ to S^6 is called *almost complex* if the differential of f is complex linear, i.e., $df_x(iv) = J_x(df_x(v)) = x \cdot df_x(v)$. It is known that ([11]) a surface Σ is an almost complex curve in S^6 if and only if the cone over Σ is an associative submanifold of $\mathbb{R}^7$.

An immersion f from a Riemann surface to S^n is called *totally isotropic* if $((\nabla_{\frac{\partial}{\partial z}})^i f_*(\frac{\partial}{\partial z}), (\nabla_{\frac{\partial}{\partial z}})^j f_*(\frac{\partial}{\partial z})) = 0$ for all $i, j \geq 0$, where $(X, Y) = \sum_{i=1}^{n+1} X_i Y_i$ is the complex bilinear form on $\mathbb{C}^{n+1}$. A surface in S^n is said to be *full* if it does not contain in any hypersphere. Bolton, Vrancken, and Woodward ([4]) used harmonic sequences to prove that if $f : \Sigma \to S^6$ is an immersed almost complex curve, then f must be one of the following:

(i) full in S^6 and totally isotropic,

*Research supported in part by NSF grant DMS-0529756.

(ii) full in S^6 and not totally isotropic,
(iii) full in some totally geodesic S^5 in S^6,
(iv) a totally geodesic S^2.

Bryant ([5]) used twistor theory to construct type (i) almost complex curves of any genus in S^6. Cones over a type (iii) almost complex curves in S^6 are special Lagrangian submanifolds, which have been studied by several authors ([8, 12, 13, 16, 15]). To state known results for type (ii) almost complex curves, we need to recall Burstall and Pedit's definition of primitive maps ([6]). Let σ be an order 6 inner automorphism of G_2 such that the fixed point set of σ is a maximal torus T^2, i.e., $\mathrm{G}_2/\mathrm{T}^2$ is a 6-symmetric space. Let $\mathfrak{h}_j$ denote the eigenspace of the complexified $d\sigma_e$ on $\mathfrak{g}_2^{\mathbb{C}} = \mathfrak{g}_2 \otimes \mathbb{C}$. A map $f : \mathbb{C} \to \mathrm{G}_2/\mathrm{T}^2$ is *primitive* if there is a lift $F : \mathbb{C} \to \mathrm{G}_2$ such that $F^{-1}F_z \in \mathfrak{h}_0 + \mathfrak{h}_{-1}$. We will call any smooth map $F : \mathbb{C} \to \mathrm{G}_2$ satisfying the condition that $F^{-1}F_z \in \mathfrak{h}_0 + \mathfrak{h}_{-1}$ a *σ-primitive G_2-frame*. Bolton, Pedit, and Woodward ([3]) proved that if $f : \Sigma \to S^6$ is a type (ii) almost complex curve, then there exists a σ-primitive G_2-frame ψ. Conversely, they show that if ψ is a σ-primitive G_2-frame, then the first column of ψ gives an almost complex curve. The equation for σ-primitive G_2-frame is an elliptic integrable system, so techniques from integrable systems can be used to study almost complex surfaces in S^6.

In this paper, we prove that if Σ is an immersed almost complex surface in S^6 such that the second fundamental form II is not zero at p_0, then there exist an open neighbor $\mathcal{O}$ of p_0 and a σ-primitive G_2-frame $\psi : \mathcal{O} \to G_2$ such that the first column is the immersion. In other words, the Gauss-Codazzi equation for the associative cones in $\mathbb{R}^7$ is the equation for σ-primitive G_2-frames. Then we use this elementary submanifold geometry set up to derive some of the known properties of almost complex curves in S^6. We also formulate the equation for S^1-symmetric almost complex curves in S^6 as a Toda type equation and use the AKS (Adler-Kostant-Symes) theory (cf. [1, 6, 2]) to construct S^1-symmetric almost complex curves.

This paper is organized as follows. We review basic properties of G_2 ([14]) in section 2, prove the existence of a σ-primitive G_2-frame on an almost complex surface with non-vanishing second fundamental form in section 3. The equation for σ-primitive G_2-frame is a system of first order PDEs for 5 complex functions, we explain in section 4 the necessary and sufficient conditions on these 5 functions corresponding to the four types of almost complex curves. In section 5, we explain how periodic Toda lattice arises from S^1-symmetric almost complex curves in S^6, and finally in section 6, we use the AKS theory to construct all S^1-symmetric almost complex curves.

2. The octonions and Lie group G_2

Let $\mathbb{H} = \mathbb{R}\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be the quaternions, where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ satisfy the condition $\mathbf{i}\cdot\mathbf{j} = \mathbf{k},\ \mathbf{j}\cdot\mathbf{k} = \mathbf{i},\ \mathbf{k}\cdot\mathbf{i} = \mathbf{j},\ \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}$. The conjugate of $a = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is $\overline{a} = a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}$. The quaternions

$\mathbb{H}$ equipped with the standard norm of $\mathbb{R}^4$ is an associative normed algebra, i.e., $\| a \cdot b \| = \| a \| \cdot \| b \|$. The octonions are defined to be $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\mathbf{e}$ with the multiplication

$$(a + b\mathbf{e}) \cdot (c + d\mathbf{e}) = (a \cdot c - \overline{d} \cdot b) + (d \cdot a + b \cdot \overline{c})\mathbf{e}$$

The octonions $\mathbb{O}$ equipped with the standard norm of $\mathbb{R}^8$ is a non-associative normed algebra. Let $\{e_1, \cdots, e_7\}$ be the standard basis of $\mathbb{R}^7$. We identify $\mathbb{R}^7$ with $\mathrm{Im}\mathbb{O}$ as follows:

$$e_1 \to \mathbf{i},\ e_2 \to \mathbf{j},\ e_3 \to \mathbf{k},\ e_4 \to \mathbf{e},\ e_5 \to \mathbf{ie},\ e_6 \to \mathbf{je},\ e_7 \to \mathbf{ke}.$$

The multiplication table of octonions is:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

The Lie group G_2 is defined by

$$\mathrm{G}_2 = \mathrm{Aut}(\mathbb{O}) = \{g \in \mathrm{GL}(\mathbb{O}) \mid g(x \cdot y) = g(x) \cdot g(y)\}$$

We list below some basic properties of the Lie group G_2 we need in this paper:

(1) Let f_1, f_2 be two orthonormal column vectors in $\mathbb{R}^7$. If $f_3 = f_1 \cdot f_2$, then f_3 is a unit vector and perpendicular to f_1, f_2. Let f_4 be a unit column vector which is perpendicular to f_1, f_2, f_3 and denote $f_5 = f_1 \cdot f_4,\ f_6 = f_2 \cdot f_4,\ f_7 = f_3 \cdot f_4$. Then $(f_1, \cdots, f_7) \in \mathrm{G}_2$ Such $\{f_1, \cdots, f_7\}$ is called a G_2-frame.
(2) Any element of G_2 can be realized by a G_2-frame.
(3) G_2 is a compact, simply-connected, simple Lie group, $\mathrm{G}_2 \subseteq \mathrm{SO}(\mathrm{Im}\mathbb{O})$, and $\dim(\mathrm{G}_2) = 14$.
(4) Let $x^1, \cdots, x^7$ be coordinates of $\mathbb{R}^7$. The 3-form $\phi(x, y, z) = (x, y \cdot z)$ can be written as

$$\phi = dx^{123} + dx^{145} - dx^{167} + dx^{246} - dx^{275} + dx^{347} - dx^{356}$$

where $dx^{jkl} = dx^j \wedge dx^k \wedge dx^l$. Then

$$\mathrm{G}_2 = \{\ g \in \mathrm{GL}(7, \mathbb{R}) \mid g^*\phi = \phi\ \}$$

(5) The Lie algebra $\mathfrak{g}_2$ of G_2 are the space of matrices

$$\begin{pmatrix} 0 & -x_2 & -x_3 & -x_4 & -x_5 & -x_6 & -x_7 \\ x_2 & 0 & -y_3 & -y_4 & -y_5 & -y_6 & -y_7 \\ x_3 & y_3 & 0 & -x_6+y_5 & -x_7-y_4 & x_4-y_7 & x_5+y_6 \\ x_4 & y_4 & x_6-y_5 & 0 & -z_5 & -z_6 & -z_7 \\ x_5 & y_5 & x_7+y_4 & z_5 & 0 & -x_2-z_7 & -x_3+z_6 \\ x_6 & y_6 & -x_4+y_7 & z_6 & x_2+z_7 & 0 & -y_3-z_5 \\ x_7 & y_7 & -x_5-y_6 & z_7 & x_3-z_6 & y_3+z_5 & 0 \end{pmatrix} \tag{2.1}$$

where $x_2, \cdots, x_7$, $y_3, \cdots, y_7$, z_5, z_6, z_7 are real numbers. To see this fact, we let $\{e_1, \cdots, e_7\}$ be the standard bases in $\mathbb{R}^7$. We have $e_3 = e_1 \cdot e_2$, $e_5 = e_1 \cdot e_4$, $e_6 = e_2 \cdot e_4$, $e_7 = (e_1 \cdot e_2) \cdot e_4$. If $A \in \mathfrak{g}_2$, then

$$A(e_j \cdot e_k) = A(e_j) \cdot e_k + e_j \cdot A(e_k)$$

So A is determined by $A(e_1), A(e_2)$ and $A(e_4)$. Let $A(e_1) = x_2 e_2 + \cdots + x_7 e_7$. Since $A \in \mathfrak{g}_2 \subset \mathfrak{so}(7)$, we can write $A(e_2) = -x_2 e_1 + y_3 e_3 + \cdots + y_7 e_7$. Then

$$\begin{aligned} A(e_3) &= A(e_1) \cdot e_2 + e_1 \cdot A(e_2) \\ &= -x_3 e_1 - y_3 e_2 + (x_6 - y_5) e_4 + (x_7 + y_4) e_5 \\ &\quad + (y_7 - x_4) e_6 - (x_5 + x_6) e_7 \end{aligned}$$

Since $A \in \mathfrak{g}_2 \subset \mathfrak{so}(7)$, we can write

$$A(e_4) = -x_4 e_1 - y_4 e_2 + (y_5 - x_6) e_3 + z_5 e_5 + z_6 e_6 + z_7 e_7$$

Similarly $A(e_5), \cdots, A(e_7)$ are determined. Thus A is a matrix of type (2.1). Conversely, any matrix of type (2.1) is a element of $\mathfrak{g}_2$.

3. σ-PRIMITIVE G_2-FRAME

Let X_2 denote the matrix defined by (2.1) with $x_2 = 1$, and all other variables being zero. The matrices $X_3, \cdots, X_7$, $Y_3, \cdots, Y_7$, Z_5, Z_6, Z_7 are defined similarly.

Let $h = \exp(\frac{\pi}{3}(Y_3 + 2Z_5))$, and $\sigma : \mathrm{G}_2 \to \mathrm{G}_2$ the order 6 inner automorphism defined by $\sigma(g) = h^{-1} g h$. The eigenspace $\mathfrak{h}_j$ with eigenvalue $\exp\left(\frac{j\pi \mathrm{i}}{3}\right)$ for the complexified $d\sigma_e$ on $\mathfrak{g}_2^{\mathbb{C}} = \mathfrak{g}_2 \otimes \mathbb{C}$ is:

$$\begin{aligned} \mathfrak{h}_0 &= \{Y_3, Z_5\} \\ \mathfrak{h}_1 &= \left\{X_2 + \mathrm{i}X_3 + \tfrac{\mathrm{i}}{2}(Z_6 + \mathrm{i}Z_7),\ Y_4 + \mathrm{i}Y_5,\ Z_6 - \mathrm{i}Z_7\right\} \\ \mathfrak{h}_2 &= \left\{X_4 + \mathrm{i}X_5 - \tfrac{\mathrm{i}}{2}(Y_6 + \mathrm{i}Y_7),\ Y_6 - \mathrm{i}Y_7\right\} \\ \mathfrak{h}_3 &= \left\{X_6 - \mathrm{i}X_7 + \tfrac{\mathrm{i}}{2}(Y_4 - \mathrm{i}Y_5),\ X_6 + \mathrm{i}X_7 - \tfrac{\mathrm{i}}{2}(Y_4 + \mathrm{i}Y_5)\right\} \\ \mathfrak{h}_4 &= \left\{X_4 - \mathrm{i}X_5 + \tfrac{\mathrm{i}}{2}(Y_6 - \mathrm{i}Y_7),\ Y_6 + \mathrm{i}Y_7\right\} \\ \mathfrak{h}_5 &= \left\{X_2 - \mathrm{i}X_3 - \tfrac{\mathrm{i}}{2}(Z_6 - \mathrm{i}Z_7),\ Y_4 - \mathrm{i}Y_5,\ Z_6 + \mathrm{i}Z_7\right\} \end{aligned}$$

Here $\{v_1, \ldots, v_m\}$ means the linear span of $v_1, \ldots, v_m$. Notice $\bar{\mathfrak{h}}_j = \mathfrak{h}_{-j}$ (we use the convention that $\mathfrak{h}_i = \mathfrak{h}_j$ if $i \equiv j \pmod 6$).

A smooth map $\psi : \mathbb{C} \to \mathrm{G}_2$ is σ-primitive if there exists $(u_0, u_{-1}) : \mathbb{C} \to \mathfrak{h}_0 + \mathfrak{h}_{-1}$ such that

$$\psi^{-1} d\psi = (u_0 + u_{-1})dz + (\bar{u}_0 + \bar{u}_{-1})d\bar{z}.$$

The flatness of $\psi^{-1}d\psi$ implies that $(u_0, u_{-1}) : \mathbb{C} \to \mathfrak{h}_0 \oplus \mathfrak{h}_{-1}$ must satisfy

$$\begin{cases} (u_0)_{\bar{z}} - (\bar{u}_0)_z = [u_0, \bar{u}_0] + [u_{-1}, \bar{u}_{-1}], \\ (u_{-1})_{\bar{z}} = [u_{-1}, \bar{u}_0]. \end{cases} \tag{3.1}$$

This system has a Lax pair

$$\theta_\lambda = \left(u_0 + \lambda^{-1} u_{-1}\right) \mathrm{d}z + (\bar{u}_0 + \lambda \bar{u}_{-1})\, \mathrm{d}\bar{z} \tag{3.2}$$

i.e., (u_0, u_{-1}) is a solution of (3.2) if and only if θ_λ is flat for all $\lambda \in \mathbb{C}\backslash\{0\}$. Note that:

(1) The Lax pair satisfies the following reality conditions:

$$\overline{\left(\theta_{1/\bar{\lambda}}\right)} = \theta_\lambda, \quad \sigma(\theta_\lambda) = \theta_{e^{\frac{\pi i}{3}}\lambda} \tag{3.3}$$

(2) $\xi(\lambda) = \sum_j \xi_j \lambda^j$ satisfies the above reality condition if and only if $\xi_j \in \mathfrak{h}_j$ and $\xi_{-j} = \bar{\xi}_j$ for all j.

The following is well-known:

Proposition 3.1. *Let $(u_0, u_{-1}) : \mathbb{C} \to \mathfrak{h}_0 \oplus \mathfrak{h}_{-1}$ be smooth maps. The following statements are equivalent:*

(1) *(u_0, u_{-1}) satisfies (3.1).*
(2) *$\theta_\lambda = \left(u_0 + \lambda^{-1}u_{-1}\right) \mathrm{d}z + (\overline{u}_0 + \lambda \overline{u}_1)\, \mathrm{d}\bar{z}$ is flat for all $\lambda \in \mathbb{C}\backslash\{0\}$, i.e., $\mathrm{d}\theta_\lambda = -\theta_\lambda \wedge \theta_\lambda$.*
(3) *$\theta_1 = (u_0 + u_{-1})\, \mathrm{d}z + (\overline{u}_0 + \overline{u}_1)\, \mathrm{d}\bar{z}$ is flat.*
(4) *There exists $\psi : \mathbb{C} \to \mathrm{G}_2$ such that $\psi^{-1}\psi_z = u_0 + u_{-1}$, i.e., ψ is a σ-primitive G_2-frame.*

Proof. The only nontrivial part is (3) $\Leftrightarrow$ (1). To see this, we decompose

$$\begin{aligned} \mathrm{d}\theta + \theta \wedge \theta \quad = \quad & (-(u_0)_{\bar{z}} + (\overline{u}_0)_z + [u_{-1}, \bar{u}_{-1})\, \mathrm{d}z \wedge \mathrm{d}\bar{z} \\ & + (-(u_{-1})_{\bar{z}} + [u_{-1}, \overline{u}_0])\, \mathrm{d}z \wedge \mathrm{d}\bar{z} \\ & + ((\bar{u}_{-1})_z + [u_0, \bar{u}_{-1}])\, \mathrm{d}z \wedge \mathrm{d}\bar{z} \end{aligned}$$

according to $\mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_{-1}$. Thus (u_0, u_{-1}) satisfies (3.1) if and only if $\mathrm{d}\theta + \theta \wedge \theta = 0$. $\square$

Suppose (u_0, u_{-1}) is a solution of (3.1). Since θ_λ is flat at $\lambda = 1$, there exists $\psi : \mathbb{C} \to G_2$ such that

$$\psi^{-1}\psi_z = u_0 + u_{-1} = \begin{pmatrix} 0 & -c & \mathrm{i}c & & & & \\ c & 0 & -a & -d & \mathrm{i}d & & \\ -\mathrm{i}c & a & 0 & -\mathrm{i}d & -d & & \\ & d & \mathrm{i}d & 0 & -b & -e+\frac{\mathrm{i}}{2}c & -\mathrm{i}e+\frac{1}{2}c \\ & -\mathrm{i}d & d & b & 0 & -\mathrm{i}e-\frac{1}{2}c & e+\frac{\mathrm{i}}{2}c \\ & & & e-\frac{\mathrm{i}}{2}c & \mathrm{i}e+\frac{1}{2}c & 0 & -a-b \\ & & & \mathrm{i}e-\frac{1}{2}c & -e-\frac{\mathrm{i}}{2}c & a+b & 0 \end{pmatrix} \tag{3.4}$$

System (3.1) written in terms of $a, \ldots, e$ is

$$\begin{cases} a_{\bar z} - \left(\overline{a}\right)_z = \mathrm{i}\left(2|c|^2 - 4|d|^2\right) \\ b_{\bar z} - \left(\overline{b}\right)_z = \mathrm{i}\left(-|c|^2 + 4|d|^2 - 4|e|^2\right) \\ c_{\bar z} = -\mathrm{i}\overline{a}c \\ d_{\bar z} = \mathrm{i}\left(\overline{a} - \overline{b}\right) d \\ e_{\bar z} = \mathrm{i}\left(\overline{a} + 2\overline{b}\right) e \end{cases} \tag{3.5}$$

Let $f_1, \ldots, f_7$ denote the columns of ψ. Then (3.4) written in columns gives

$$\begin{cases} (f_1)_z = cf_2 - icf_3, \\ (f_2)_z = -cf_1 + af_3 + d\ f_4 - id\ f_5, \\ (f_3)_z = icf_1 - af_2 + id\ f_4 + d\ f_5, \\ (f_4)_z = -d\ f_2 - id\ f_3 + bf_5 + (e - \frac{ic}{2})f_6 + (ie - \frac{c}{2})f_7, \\ (f_5)_z = id\ f_2 - d\ f_3 - bf_4 + (ie + \frac{c}{2})f_6 - (e + \frac{ic}{2})f_7, \\ (f_6)_z = (-e + \frac{i}{2}c)f_4 - (ie + \frac{c}{2})f_5 + (a+b)f_7, \\ (f_7)_z = (-ie + \frac{c}{2})f_4 + (e + \frac{ic}{2})f_5 - (a+b)f_6. \end{cases} \tag{3.6}$$

4. Associative cones and almost complex curves

The following well-known Proposition relates almost complex curves to associative cones:

Proposition 4.1. *([11]) Let Σ be a 2-dimensional surface in S^6, and $\mathrm{C}(\Sigma) = \{tx \mid t > 0, x \in M\}$ the cone of Σ in $\mathbb{R}^7$. Then $\mathrm{C}(\Sigma)$ is an associative submanifold in $\mathbb{R}^7$ if and only if Σ is a almost complex curve in S^6.*

Proof. Let $\{e_1, e_2\}$ be an orthonormal basis of $T_x\Sigma$. Then $\{x, e_1, e_2\}$ is an orthonormal basis of $T_x\mathrm{C}(\Sigma)$. Lemma follows from the fact that $\mathbb{R}\{\mathbf{1}, x, e_1, e_2\}$ is an associative subalgebra if and only if $x \cdot e_1 = e_2$. □

So the study of associative cones in $\mathbb{R}^7$ reduces to the study of almost complex curves in S^6.

Since associative cones are calibrated by the 3-form ϕ, they are minimal. But a cone $C(\Sigma)$ in $\mathbb{R}^7$ is minimal if and only if Σ is minimal in S^6, so almost complex curves in S^6 are minimal.

Theorem 4.2. *([3]) If $\psi = (f_1, \cdots, f_7) : \mathbb{C} \to \mathrm{G}_2$ satisfies*

$$\psi^{-1}\psi_z \in \mathfrak{h}_0 \oplus \mathfrak{h}_{-1} \tag{4.1}$$

Then $f_1 : \mathbb{C} \to \mathrm{S}^6$ is almost complex. Conversely, if $f : \mathbb{C} \to S^6$ is a type (ii) almost complex curve, i.e., f is full and not totally isotropic, then there exists a σ-primitive map $\psi : \mathbb{C} \to G_2$ such that the first column of ψ is f.

The first part of the above theorem is easy to see: Write $\psi = (f_1, \ldots, f_7)$, and

$$\psi^{-1}\psi_z = u_0 + u_{-1}.$$

Then $u_0 + u_{-1}$ is given by (3.4), so

$$(f_1)_z = cf_2 - \mathrm{i}cf_3$$

By the definition of almost complex structure J on S^6, we have

$$J\,(f_1)_z = f_1 \cdot (f_1)_z = cf_3 + \mathrm{i}cf_2 = \mathrm{i}\,(f_1)_z$$

So f_1 is almost complex.

Next we prove that a σ-primitive G_2-frame exists on any almost complex curve in S^6 with non-vanishing second fundamental forms.

Theorem 4.3. *Suppose $f_1 : \Sigma \to S^6$ is an almost complex curve such that the second fundamental form* II *is non-zero at some $p_0 \in \Sigma$. Then there exists a neighborhood $\mathcal{O}$ of p_0 and a σ-primitive G_2-frame $\psi = \{f_1, \cdots, f_7\}$ on $\mathcal{O}$ such that f_2 and f_3 are tangent to the immersion, $\psi^{-1}\psi_z$ is given by* (3.4) *in terms of* 5 *functions $a, \ldots, e$, and* (3.5) *is the Gauss-Codazzi equation for f_1. Moreover, the first and second fundamental forms of f_1 are*

$$\begin{aligned} &\mathrm{I} = 2|c|^2|dz|^2, \\ &\mathrm{II}\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) = 2cd(f_4 - \mathrm{i}f_5), \end{aligned}$$

and the normal connection is given by the lower 4×4 matrices (3.4).

Proof. Locally we can choose orthonormal tangent frame $\{f_2, f_3\}$ such that $f_3 = f_1 \cdot f_2$. Let f_4 be an arbitrary unit vector such that $f_4 \perp \mathrm{span}_{\mathbb{R}}\{f_1, f_2, f_3\}$. Then we have a G_2-frame $\psi = \{f_1, \cdots, f_7\}$ where $f_5 = f_1 \cdot f_4, f_6 = f_2 \cdot f_4, f_7 = f_3 \cdot f_4$. Therefore we obtain a $\mathfrak{g}_2$-valued flat connection 1-form $\omega = (\omega_{ij}) = \psi^{-1}\mathrm{d}\psi$.

Write

$$\mathrm{d}f_1 = f_2 \otimes \theta_2 + f_3 \otimes \theta_3,$$

where θ_j is the dual 1-form of f_j for $j = 2, 3$. Therefore

$$\begin{aligned} &\omega_{21} = \theta_2, \quad \omega_{31} = \theta_3 \\ &\omega_{\alpha 1} = 0, \quad 4 \le \alpha \le 7 \end{aligned}$$

Since ω is $\mathfrak{g}_2$-valued, we have

$$\omega_{43} = -\omega_{52},\ \omega_{53} = \omega_{42},\ \omega_{63} = \omega_{72},\ \omega_{73} = -\omega_{62}$$

Let

$$\omega_{52} = a_2\theta_2 + a_3\theta_3,\ \omega_{62} = b_2\theta_2 + b_3\theta_3$$

It follows from the flatness of (ω_{ij}) that

$$d\omega_{\alpha 1} + \sum_{j=1}^{7} \omega_{\alpha j} \wedge \omega_{j1} = 0,\ (\alpha = 4, 5),$$

so we have

$$\begin{aligned} (a_2\theta_2 + a_3\theta_3) \wedge \theta_2 + \omega_{42} \wedge \theta_3 &= 0 \\ \omega_{42} \wedge \theta_2 - (a_2\theta_2 + a_3\theta_3) \wedge \theta_3 &= 0 \end{aligned}$$

Thus

$$\omega_{53} = \omega_{42} = a_3\theta_2 - a_2\theta_3$$

Similarly,

$$\omega_{63} = \omega_{72} = b_3\theta_2 - b_2\theta_3$$

Then the second fundamental form of immersion is given by

$$\begin{aligned} \mathrm{II} &= \sum_{\alpha=4}^{7} f_\alpha \otimes (\omega_{\alpha 2} \otimes \theta_2 + \omega_{\alpha 3} \otimes \theta_3) \\ &= v_1 \otimes (\theta_2 \otimes \theta_2 - \theta_3 \otimes \theta_3) - v_2 \otimes (\theta_2 \otimes \theta_3 + \theta_3 \otimes \theta_2) \end{aligned}$$

where $v_1 = a_3 f_4 + a_2 f_5 + b_3 f_7 + b_2 f_6$ and $v_2 = a_2 f_4 - a_3 f_5 + b_2 f_7 - b_3 f_6$. Note that

$$(v_1, v_1) = (v_2, v_2), \quad (v_1, v_2) = 0$$

Since $\mathrm{II}(p_0) \neq 0$, there exists a neighborhood U of p such that v_1 and v_2 are nonzero. Let $\widetilde{f}_j = f_j,\ j = 1, 2, 3,$

$$\widetilde{f}_4 = \frac{v_1}{||v_1||}$$

and

$$\widetilde{f}_5 = \widetilde{f}_1 \cdot \widetilde{f}_4, \quad \widetilde{f}_6 = \widetilde{f}_2 \cdot \widetilde{f}_4, \quad \widetilde{f}_7 = \widetilde{f}_3 \cdot \widetilde{f}_4$$

Then $\widetilde{\psi} = \{\widetilde{f}_1, \cdots, \widetilde{f}_7\}$ is a G_2-frame, and a computation using the octonion multiplication implies that $\tilde{f}_5 = v_2/||v_2||$. Let $\widetilde{\omega} = (\widetilde{\omega}_{ij}) = \widetilde{\psi}^{-1} \mathrm{d}\widetilde{\psi}$. Since $(\mathrm{II}, \tilde{f}_6) = (\mathrm{II}, \tilde{f}_7) = 0$, we have

$$\widetilde{\omega}_{62} = \widetilde{\omega}_{63} = \widetilde{\omega}_{72} = \widetilde{\omega}_{73} \equiv 0.$$

So $\tilde{\omega}$ lies in $\mathfrak{h}_0 + \mathfrak{h}_1 + \mathfrak{h}_{-1}$, where $\mathfrak{h}_j$ is the eigenspace of $d\sigma$ on $\mathfrak{g}_2 \otimes \mathbb{C}$ with eigenvalue $e^{\frac{2\pi j i}{6}}$. Or equivalently, $\psi^{-1}\psi_z$ is of the form (3.4), i.e., ψ is a σ-primitive G_2-frame. In particular, this shows that the Gauss-Codazzi equation for almost complex curves is (3.5). It follows from (3.6) and a computation that the two fundamental forms for f_1 are given as in the Theorem. □

As a consequence of the Fundamental Theorem of submanifolds in space forms and the above theorem, we get

Corollary 4.4. *Every simply connected immersed almost complex curve in* (S^6, J) *with non-vanishing second fundamental form has a* σ*-primitive* G_2*-frame such that the first column is the immersion. Conversely, the first column of a* σ*-primitive* G_2*-frame is an almost complex surface in* S^6.

Next, we use Theorem 4.3 to give conditions on $a, \ldots, e$ to determine the four types of almost complex curves mentioned in the introduction.

Corollary 4.5. *Let* $(a, \ldots, e)$ *be a solution of* (3.5), ψ *a solution of* (3.4), *and* f_1 *the first column of* ψ. *Then* f_1 *is almost complex in* S^6 *and is*

(i) *full in* S^6 *and totally isotropic if and only if* $e \equiv 0$ *and* $d \neq 0$,
(ii) *full in* S^6 *and not totally isotropic if and only if* $de \neq 0$,
(iii) *full in* S^5 *if and only if* $de \neq 0$ *and* $a + b \equiv 0$,
(iv) *totally geodesic two sphere if and only if* $d \equiv 0$, *i.e.,* $\mathrm{II} \equiv 0$.

Moreover, the cone over the curve of type (iii) is a special Lagrangian cone in $\mathbb{R}^6$ *with the appropriate complex structure.*

Proof. The first fundamental form is positive definite, so $c \neq 0$. A surface is full then II can not be zero, so $d \neq 0$. Let ψ satisfy $\psi^{-1}d\psi = (u_0 + u_{-1})dz + (\bar{u}_0 + \bar{u}_{-1})d\bar{z}$, and f_1 denote the first column of ψ, where $u_0 + u_{-1} \in \mathfrak{h}_0 + \mathfrak{h}_{-1}$ is given by (3.4). Then f_1 is almost complex. Use (3.6) and a direct computation to see that

$$((\nabla_{\frac{\partial}{\partial z}})^2 f_*(\frac{\partial}{\partial z}), \ \ (\nabla_{\frac{\partial}{\partial z}})^2 f_*(\frac{\partial}{\partial z})) = -32\mathrm{i}c^3 d^2 e,$$

where $(Y, Z) = \sum_j y_j z_j$ is the complex bilinear form on $\mathbb{C}^7$. If f is totally isotropic, then

$$((\nabla_{\frac{\partial}{\partial z}})^i f_*(\frac{\partial}{\partial z}), \ \ (\nabla_{\frac{\partial}{\partial z}})^j f_*(\frac{\partial}{\partial z})) = 0$$

for all other $0 \leq i, j \leq 2$, so $e = 0$.

Next we prove that if an almost complex curve is of type (iii), then $a+b \equiv 0$. Since there is a constant unit normal vector field on the curve, there exists real functions λ_i $(4 \leq i \leq 7)$ on the curve such that this normal vector is $\sum_{i=4}^7 \lambda_i f_i$. Then

$$(\sum_{i=4}^7 \lambda_i f_i)_z = \sum_{i=4}^7 (\lambda_i)_z f_i + \sum_{i=4}^7 \lambda_i (f_i)_z$$

$$= \sum_{i=4}^7 (\lambda_i)_z f_i + \lambda_4[-df_2 - idf_3 + bf_5 + (e - ic/2)f_6 + (ie - c/2)f_7] + \cdots = 0.$$

So the coefficient of f_i must be zero for $2 \leq i \leq 7$. Since $d \neq 0$, it implies that $\lambda_4 = 0$ and $\lambda_5 = 0$. The coefficients for f_6 and f_7 are $(\lambda_6)_z - (a + b)$ and $(\lambda_7)_z + (a + b)$ respectively. Therefore $(\lambda_6 + \lambda_7)_z = 0$, i.e., $\lambda_6 + \lambda_7$ is anti-holomorphic. Since $\lambda_6 + \lambda_7$ is also real, it must be a constant. Finally both λ_6 and λ_7 have to be constant because their square sum is 1. Thus $a + b = (\lambda_6)_z = 0$.

Conversely, if $a+b\equiv 0$, then the system (3.5) implies that

$$c_{\bar z}=-\mathrm{i}\overline{a}c,\quad e_{\bar z}=-\mathrm{i}\overline{a}e$$

and $\mathrm{i}\left(|c|^2-4|e|^2\right)=(a+b)_{\bar z}-\left(\overline{a}+\overline{b}\right)_z=0$. Let $\alpha=\frac{c}{2e}$. Then $\alpha_{\bar z}=0$ and $|\alpha|=1$. So $\alpha\in\mathrm{S}^1$ is a constant and $\beta=\frac{-1+\mathrm{i}\alpha}{-\mathrm{i}+\alpha}$ is a real constant. It follows from (3.6) that $(f_6-\beta f_7)_{\bar z}=0$. Thus $n=\frac{1}{\sqrt{1+\beta^2}}(f_6-\beta f_7)$ is a unit constant normal vector. So the image of the immersion lies in the hyperplane V which is orthogonal to n. Note $J(x)=n\cdot x$ defines a complex structure on the hyperplane V and $J(f_1)=\frac{1}{\sqrt{1+\beta^2}}(\beta f_6+f_7)$, $J(f_2)=\frac{1}{\sqrt{1+\beta^2}}(f_4+\beta f_5)$, and $J(f_3)=\frac{1}{\sqrt{1+\beta^2}}(-\beta f_4-f_5)$. Thus $J(\mathrm{span}_{\mathbb{R}}\{f_1,f_2,f_3\})=\mathrm{span}_{\mathbb{R}}\{f_4,f_5,\beta f_6+f_7\}$, so the cone over the image of f_1 is Lagrangian in $\left(\mathbb{R}^6,J\right)$. We know it is minimal, so by Proposition 2.17 of [11] that it is θ-special Lagrangian for some θ. $\square$

Next we use Theorem 4.3 to give a proof of one of Bryant's results on almost complex curves in S^6. First recall that the 5-dimensional complex quadric Q_5 is defined by

$$Q_5=\{[z_1:\cdots:z_7]\in\mathbb{C}\mathrm{P}^6\mid z_1^2+\cdots+z_7^2=0\}.$$

Theorem 4.6. [5] *If $f:\Sigma\to\mathrm{S}^6$ is a totally isotropic almost complex curve that is not totally geodesic, then it can be lifted to a horizontal holomorphic map to Q_5.*

Proof. Let $\psi=(f_1,\cdots,f_7):\Sigma\to\mathrm{G}_2$ denote the σ-primitive G_2-frame obtained in Theorem 4.3. So $\psi^{-1}\psi_z$ is of the form (3.4). Let $\Phi:\Sigma\to Q_5$ be the map defined by

$$\Phi=[f_6+if_7]$$

Clearly Φ is well-defined and is independent of choice of the frame. By (3.6), we have

$$(f_6+\mathrm{i}f_7)_{\bar z}=-2\bar e(f_4-\mathrm{i}f_5)-\mathrm{i}(\bar a+\bar b)(f_6+\mathrm{i}f_7)$$

But we have shown in Corollary 4.5 that if f is totally isotropic then $e=0$, so Φ is holomorphic. $\square$

5. S^1-symmetric solutions and periodic Toda lattice

By the maximal torus theorem, given $A\in\mathcal{G}_2$, there exists $k\in\mathrm{G}_2$ and real numbers λ_1,λ_2 such that $A=k^{-1}(\lambda_1Y_3+\lambda_2Z_5)k$. Note

$$\lambda_1Y_3+\lambda_2Z_5=\begin{pmatrix}0&&&&&&\\&&-\lambda_1&&&&\\&\lambda_1&&&&&\\&&&&-\lambda_2&&\\&&&\lambda_2&&&\\&&&&&&\lambda_3\\&&&&&-\lambda_3&\end{pmatrix}$$

where $\lambda_3 = -(\lambda_1 + \lambda_2)$. We say $A = k^{-1}(\lambda_1 Y_3 + \lambda_2 Z_5)k$ is *rational* if λ_1, λ_2 are linearly dependent over the rationals. It is easy to see that A is rational if and only if $\{\exp(sA) \mid s \in \mathbb{R}\}$ is periodic.

To construct a S^1-symmetric almost complex curve in S^6, we need to construct $\psi = e^{As} g(t)$ with rational A and $g(t) \in \mathrm{G}_2$ such that

$$\psi^{-1}\psi_z = u_0 + u_{-1} \in \mathfrak{h}_0 + \mathfrak{h}_1,$$

where $z = s + it$ and $u_0 + u_{-1}$ is given by (3.4) and a, b, c, d, e are complex valued functions of t only. A simple computation gives

$$\psi^{-1}\mathrm{d}\psi = \left(g^{-1}Ag\right)\mathrm{d}s + \left(g^{-1}g_t\right)\mathrm{d}t.$$

The flatness of $\psi^{-1}\mathrm{d}\psi$ implies that

$$\left(g^{-1}Ag\right)_t = \left[g^{-1}Ag, g^{-1}g_t\right]. \tag{5.1}$$

Write $a = a_1 + \mathrm{i}a_2,\ b = b_1 + \mathrm{i}b_2,\ \ldots,\ e = e_1 + \mathrm{i}e_2$ in real and imaginary part, and $c = r_1 e^{\mathrm{i}\beta_1},\ d = r_2 e^{\mathrm{i}\beta_2},\ e = r_3 e^{\mathrm{i}\beta_3}$ in polar coordinates. Since $\psi^{-1}\psi_s = g^{-1}Ag = \psi^{-1}\psi_z + \psi^{-1}\psi_{\bar z}$, $\psi^{-1}\psi_t = g^{-1}g_t = \mathrm{i}\ (\psi^{-1}\psi_z - \psi^{-1}\psi_{\bar z})$, and $\psi^{-1}\psi_z$ is given by (3.4), we have

$$g^{-1}Ag = \begin{pmatrix} 0 & -2c_1 & -2c_2 & & & & \\ 2c_1 & 0 & -2a_1 & -2d_1 & -2d_2 & & \\ 2c_2 & 2a_1 & 0 & 2d_2 & -2d_1 & & \\ & 2d_1 & -2d_2 & 0 & -2b_1 & -2e_1 - c_2 & 2e_2 + c_1 \\ & 2d_2 & 2d_1 & 2b_1 & 0 & 2e_2 - c_1 & 2e_1 - c_2 \\ & & & 2e_1 + c_2 & -2e_2 + c_1 & 0 & -2a_1 - 2b_1 \\ & & & -2e_2 - c_1 & -2e_1 + c_2 & 2a_1 + 2b_1 & 0 \end{pmatrix},$$

$$g^{-1}g_t = \begin{pmatrix} 0 & 2c_2 & -2c_1 & & & & \\ -2c_2 & 0 & 2a_2 & 2d_2 & -2d_1 & & \\ 2c_1 & -2a_2 & 0 & 2d_1 & 2d_2 & & \\ & -2d_2 & -2d_1 & 0 & 2b_2 & 2e_2 - c_1 & 2e_1 - c_2 \\ & 2d_1 & -2d_2 & -2b_2 & 0 & 2e_1 + c_2 & -2e_2 - c_1 \\ & & & -2e_2 + c_1 & -2e_1 - c_2 & 0 & 2a_2 + 2b_2 \\ & & & -2e_1 + c_2 & 2e_2 + c_1 & -2a_2 - 2b_2 & 0 \end{pmatrix}.$$

System (5.1) written in a, b, r_i, β_i gives the following two separable systems

$$\begin{cases} \dot a_1 = 2r_1^2 - 4r_2^2, \\ \dot b_1 = -r_1^2 + 4r_2^2 - 4r_3^2, \\ \dot r_1 = -2a_1 r_1, \\ \dot r_2 = 2(a_1 - b_1)r_2, \\ \dot r_3 = 2(a_1 + 2b_1)r_3, \end{cases} \qquad \begin{cases} \dot\beta_1 = 2a_2, \\ \dot\beta_2 = -2a_2 + 2b_2, \\ \dot\beta_3 = -2a_2 - 4b_2. \end{cases}$$

So we may assume that $a_2 = b_2 = \beta_1 = \beta_2 = \beta_3 = 0$, i.e.,

$$a_2 = b_2 = c_2 = d_2 = e_2 = 0.$$

Substitute these conditions to the matrix formulas for $g^{-1}Ag$ and $g^{-1}g_t$ to get

$$P := g^{-1}Ag = \begin{pmatrix} 0 & -2c_1 & & & & & \\ 2c_1 & 0 & -2a_1 & -2d_1 & & & \\ & 2a_1 & 0 & & -2d_1 & & \\ & 2d_1 & & 0 & -2b_1 & -2e_1 & c_1 \\ & & 2d_1 & 2b_1 & 0 & -c_1 & 2e_1 \\ & & & 2e_1 & c_1 & 0 & -2a_1-2b_1 \\ & & & -c_1 & -2e_1 & 2a_1+2b_1 & 0 \end{pmatrix},$$

$$Q := g^{-1}g_t = \begin{pmatrix} 0 & & -2c_1 & & & & \\ & 0 & & & -2d_1 & & \\ 2c_1 & & 0 & 2d_1 & & & \\ & & -2d_1 & 0 & & -c_1 & 2e_1 \\ & 2d_1 & & & 0 & 2e_1 & -c_1 \\ & & & c_1 & -2e_1 & 0 & \\ & & & -2e_1 & c_1 & & 0 \end{pmatrix}$$

Since $\psi^{-1}\psi_z = u_0 + u_{-1} \in \mathfrak{h}_0 + \mathfrak{h}_{-1}$, $P = u_0 + \bar{u}_0 + u_{-1} + \bar{u}_{-1}$ and $Q = -i(u_0 - \bar{u}_0 + u_{-1} - \bar{u}_{-1})$. By assumption that $a, b, \ldots, e$ are real, so $u_0 = \bar{u}_0$, and

$$P = 2u_0 + u_{-1} + \bar{u}_{-1}, \quad Q = i(u_{-1} - \bar{u}_{-1}), \tag{5.2}$$

where

$$\begin{cases} u_0 = a_1 Y_3 + b_1 Z_5 \quad \in \mathfrak{h}_0 \cap \mathfrak{g}_2, \\ u_{-1} = c_1(X_2 - \frac{Z_7}{2}) + i(X_3 + \frac{Z_6}{2}) + d_1(Y_4 + iY_5) + e_1(Z_6 - iZ_7) \quad \in \mathfrak{h}_{-1}. \end{cases}$$

Thus we have

Proposition 5.1. *Suppose* $(u_0, u_{-1}) : \mathbb{R} \to (\mathfrak{h}_0 \cap \mathfrak{g}_2) \times \mathfrak{h}_{-1}$ *satisfies*

$$(2u_0 + u_{-1} + \bar{u}_{-1})_t = [2u_0 + u_{-1} + \bar{u}_{-1},\ i(u_{-1} - \bar{u}_{-1})], \tag{5.3}$$

and there exist a constant $A \in (\mathfrak{h}_0 \cap \mathfrak{g}_2) + \mathfrak{h}_{-1}$ *and* $g : \mathbb{R} \to \mathrm{G}_2$ *such that*

$$\begin{cases} g^{-1}Ag = 2u_0 + u_{-1} + \bar{u}_{-1}, \\ g^{-1}g_t = u_{-1} - \bar{u}_{-1}. \end{cases} \tag{5.4}$$

Then $f(s,t) = e^{As}g(t)$ *is an almost complex curve in* S^6*. Moreover,* f *is* S^1*-symmetric if and only if* A *is rational, and is doubly periodic if and only if* A *is rational and* g *is periodic.*

Define v_1, v_2, v_3 by

$$\begin{cases} e^{2v_1} = c_1^2 \\ e^{2(v_2 - v_1)} = d_1^2 \\ e^{2(v_3 - v_2)} = e_1^2. \end{cases}$$

Then a_1, b_1, v_1, v_2, v_3 satisfy

$$\begin{cases} \dot{a}_1 = 2e^{2v_1} - 4e^{2(v_2-v_1)}, \\ \dot{b}_1 = -e^{2v_1} + 4e^{2(v_2-v_1)} - 4e^{2(v_3-v_2)}, \\ \dot{v}_1 = -2a_1, \\ \dot{v}_2 = -2b_1, \\ \dot{v}_3 = 2(a_1+b_1). \end{cases} \tag{5.5}$$

Clearly, $(v_1+v_2+v_3)_t = 0$. Moreover, v_1, v_2, v_3 satisfy

$$\begin{cases} \ddot{v}_1 = -4e^{2v_1} + 8e^{2(v_2-v_1)}, \\ \ddot{v}_2 = 2e^{2v_1} - 8e^{2(v_2-v_1)} + 8e^{2(v_3-v_2)}, \\ \ddot{v}_3 = 2e^{2v_1} - 8e^{2(v_3-v_2)}. \end{cases}$$

These are equivalent to the periodic Toda lattice equations of G_2-type.

If $a_1+b_1=0$, i.e., the type (iii) case, then $\dot{a}_1+\dot{b}_1 = e^{2v_1} - 4e^{2(v_3-v-2)} = 0$, $\dot{v}_1+\dot{v}_2 = \dot{v}_3 = 0$, so there is a positive constant C_1 such that:

$$e^{2(v_1+v_2)} = 4e^{2v_3} = C_1.$$

Then v_1 satisfies

$$\ddot{v}_1 + 4e^{2v_1} - 8C_1e^{-4v_1} = 0$$

Multiply $\dot{v}_1$ to both sides and integrating once to get

$$(\dot{v}_1)^2 + 4e^{2v_1} + 4C_1e^{-4v_1} = 4C_2,$$

where C_2 is a positive constant. Let $y = e^{2v_1} = r_1^2$. Then the above equation becomes

$$(\dot{y})^2 = -16y^3 + 16C_2y^2 - 16C_1.$$

One can verify easily that $4C_2^3 \geq 27C_1$. Therefore this equation has three real constant solutions $\Gamma_1, \Gamma_2, \Gamma_3$. Let us label these solutions so that $\Gamma_1 < 0 < \Gamma_2 \leq \Gamma_3$. Then we can rewrite the previous equation as

$$(\dot{y})^2 = -16\,(y-\Gamma_1)\,(y-\Gamma_2)\,(y-\Gamma_3)$$

Haskins ([12]) showed that this equation has the following solution:

$$y = \Gamma_3 - (\Gamma_3-\Gamma_2)\,\mathrm{sn}^2(B_1t+B_2, B_3)$$

where B_2 is a constant determined by the initial condition of y,

$$B_1^2 = 4(\Gamma_3-\Gamma_1), \qquad B_3^2 = \frac{\Gamma_3-\Gamma_2}{\Gamma_3-\Gamma_1}$$

and sn is the Jacobi elliptic sn-noidal function. Recall that $\mathrm{sn}(t,k)$ is defined to be the unique solution of the equation

$$\dot{z}^2 = \left(1-z^2\right)\left(1-k^2z^2\right)$$

with $z(0) = 0, \dot{z}(0) = 1$, where $0 \le k \le 1$. It is straightforward to see from this definition that $\mathrm{sn}(t, 0) = \sin\, t$ and $\mathrm{sn}(t, 1) = \tanh\, t$. The period of $\mathrm{sn}(t, k)$ is given by

$$\int_0^{2\pi} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

Thus y is a periodic function, so are a_1, b_1, v_1, v_2. They all have same period denoted by T.

In fact, Haskins proved in [12] that not only (5.3) has a periodic solution but he also proved that the solution g of (5.4) is also periodic for some rational A. So he proved the existence of infinitely many S^1-symmetric type (iii) almost complex curves (hence infinitely many special Lagrangian cones in $\mathbb{C}^3$).

6. S^1-symmetric solutions and loop group factorization

The first equation of (5.4) implies that the solution $2u_0(t)+u_{-1}(t)+\bar{u}_{-1}(t)$ must lie in the same conjugate class for all t, and there is g solves (5.4). Although these conditions seem to be extra conditions for solutions of (5.3), we will see below that (5.3) has a Lax pair and is a Toda type equation, and hence the AKS theory implies that if (u_0, u_{-1}) is a solution of (5.3) then there exists g satisfies (5.4) automatically.

Set $P = 2u_0 + u_{-1} + \bar{u}_{-1}$ and $Q = i(u_{-1} - \bar{u}_{-1})$ as in (5.2). Then (5.4) is $P_t = [P, Q]$, or equivalently, $iP_t = [iP, Q]$, i.e.,

$$(v_0 + v_{-1} - \bar{v}_{-1})_t = [v_0 + v_{-1} - \bar{v}_{-1}, v_{-1} + \bar{v}_{-1}], \tag{6.1}$$

where $v_0 \in \mathfrak{h}_0 \cap i\mathfrak{g}_2$ and $v_{-1} \in \mathfrak{h}_{-1}$.

Equation (6.1) has a Lax pair

A simple calculation shows that (v_0, v_1) satisfies (6.1) if and only if

$$(v_0 + v_{-1}\lambda^{-1} - \bar{v}_{-1}\lambda)_t = [v_0 + v_{-1}\lambda^{-1} - \bar{v}_{-1}\lambda,\ v_{-1}\lambda^{-1} + \bar{v}_{-1}\lambda] \tag{6.2}$$

holds for all $\lambda \in \mathbb{C} \setminus \{0\}$. Here $v_0 \in \mathfrak{h}_0$ is pure imaginary and $v_{-1} \in \mathfrak{h}_{-1}$.

Results from the Adler-Kostant-Symes (AKS) Theory (cf. [1, 6, 2])

Let G be a group, G_+, G_- subgroups of G such that the multiplication map $G_+ \times G_- \to G$ defined by $(g_+, g_-) \to g_+g_-$ is a bijection. So $\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_-$ as direct sum of vector subspaces. Suppose $\mathcal{G}$ admits a non-degenerate, ad-invariant bilinear form $(\ ,)$. Let

$$\mathcal{G}_+^\perp = \{y \in \mathcal{G} \mid (y, x) = 0\ \forall\ x \in \mathcal{G}_+\}, \tag{6.3}$$

and π_+ denote the projection of $\mathcal{G}$ onto $\mathcal{G}_+$ with respect to the decomposition $\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_-$. Suppose $M \subset \mathcal{G}_+^\perp$ is invariant under the flow

$$\frac{dx}{dt} = [x(t), \pi_+(x(t))].$$

Given $x_0 \in M$, consider the following ODE:

$$\begin{cases} \frac{dx}{dt} = [x(t), \pi_+(x(t))], \\ x(0) = x_0. \end{cases} \tag{6.4}$$

The AKS theory gives a method to solve the initial value problem (6.4) via factorizations as follows:

(i) Find the one-parameter subgroup $f(t)$ for x_0, i.e., solve $f^{-1}f_t = x_0$ with $f(0) = e$.
(ii) Factor $f(t) = f_+(t)f_-(t)$ with $f_\pm(t) \in G_\pm$.
(iii) Set $x(t) = f_+(t)^{-1}x_0f_+(t)$. Then $x(t)$ is the solution for the initial value problem (6.4). Moreover, $f_+^{-1}(f_+)_t = \pi_+(x(t))$.

If $G = SL(n, \mathbb{R})$, $G_+ = SO(n)$, $G_- =$ the subgroup of upper triangular matrices, and M is the space of all tri-diagonal matrices in $sl(n, \mathbb{R})$, then ODE (6.4) is the standard Toda lattice. So we call a system obtained from a factorization a *Toda type equation*.

Equation (6.1) **is of Toda type**

Let $L(\mathrm{G}_2^{\mathbb{C}})$ denote the group of smooth loops from S^1 to $\mathrm{G}_2^{\mathbb{C}}$ satisfying the reality condition $\overline{g(\bar{\lambda}^{-1})} = g(\lambda)$, $L_+(\mathrm{G}_2^{\mathbb{C}})$ the subgroup of $g \in L(\mathrm{G}_2^{\mathbb{C}})$ with $g(\lambda) \in \mathrm{G}_2$ for all $\lambda \in S^1$, and $L_-(\mathrm{G}_2^{\mathbb{C}})$ denote the subgroups of $f \in L(\mathrm{G}_2^{\mathbb{C}})$ that can be extended to a holomorphic maps inside S^1 such that $f(0) = e$ the identity of G. Pressely and Segal proved in [17] an analogue of the Iwasawa decomposition of simple Lie groups for loop groups:

Theorem 6.1. (Iwasawa loop group factorization Theorem [17, 10])
The multiplication map $L_+(\mathrm{G}_2^{\mathbb{C}}) \times L_-(\mathrm{G}_2^{\mathbb{C}}) \to L(\mathrm{G}_2^{\mathbb{C}})$ is a diffeomorphism. In particular, given $g \in L(\mathrm{G}_2^{\mathbb{C}})$, we can factor $g = g_+g_-$ uniquely with $g_\pm \in L_\pm(\mathrm{G}_2^{\mathbb{C}})$.

Note that

$$\hat{\sigma}(g)(\lambda) = \sigma(g(e^{-\frac{\pi i}{3}}\lambda))$$

defines an automorphism of $L(\mathrm{G}_2^{\mathbb{C}})$. Let $L^\sigma(\mathrm{G}_2^{\mathbb{C}})$ and $L_\pm^\sigma(\mathrm{G}_2^{\mathbb{C}})$ denote the subgroups fixed by $\hat{\sigma}$ of $L(\mathrm{G}_2^{\mathbb{C}})$ and $L_\pm(\mathrm{G}_2^{\mathbb{C}})$ respectively. Then we have

Corollary 6.2. *If $g \in L^\sigma(\mathrm{G}_2^{\mathbb{C}})$ and $g = g_+g_-$ with $g_\pm \in L_\pm(\mathrm{G}_2^{\mathbb{C}})$, then $g_\pm \in L_\pm^\sigma(\mathrm{G}_2^{\mathbb{C}})$.*

Let B denote the Borel subgroup of $\mathrm{G}_2^{\mathbb{C}}$ such that the Iwasawa decomposition is $\mathrm{G}_2^{\mathbb{C}} = G_2B$, and $\mathfrak{g}_2^{\mathbb{C}} = \mathfrak{g}_2 + \mathfrak{b}$ at the Lie algebra level. It is easier to write down the factorization at the Lie algebra level:

$$\mathcal{L}^\sigma(\mathfrak{g}_2^{\mathbb{C}}) = \mathcal{L}_+^\sigma(\mathfrak{g}_2^{\mathbb{C}}) + \mathcal{L}_-^\sigma(\mathfrak{g}_2^{\mathbb{C}}), \tag{6.5}$$

where

$$\mathcal{L}^\sigma(\mathfrak{g}_2^{\mathbb{C}}) = \{\xi = \sum_{j\in\mathbb{Z}} \xi_j\lambda^j \mid \xi_j \in \mathfrak{g}_2^{\mathbb{C}}, \xi_j \in \mathfrak{h}_j\},$$
$$\mathcal{L}_+^\sigma(\mathfrak{g}_2^{\mathbb{C}}) = \{\xi = \sum_{j\in\mathbb{Z}} \xi_j\lambda^j \in \mathcal{L}^\sigma(\mathfrak{g}_2^{\mathbb{C}}) \mid \xi_{-j} = \bar{\xi}_j\},$$
$$\mathcal{L}_-^\sigma(\mathfrak{g}_2^{\mathbb{C}}) = \{\xi = \sum_{j\geq 0} \xi_j\lambda^j \in \mathcal{L}^\sigma(\mathfrak{g}_2^{\mathbb{C}}) \mid \xi_0 \in \mathfrak{b}\}.$$

Let $\pi_{\mathfrak{g}_2}$ and $\pi_{\mathfrak{b}}$ denote the projections of $\mathfrak{g}_2^{\mathbb{C}}$ onto $\mathfrak{g}_2$ and $\mathfrak{b}$ respectively, and $\pi_\pm$ the projections of $\mathcal{L}^\sigma(\mathfrak{g}_2^{\mathbb{C}})$ onto $\mathcal{L}_\pm^\sigma(\mathfrak{g}_2^{\mathbb{C}})$ with respect to the decomposition (6.5). Then for $\xi = \sum_j \xi_j\lambda^j$,

$$\pi_+(\xi) = \pi_{\mathfrak{g}_2}(\xi_0) + \sum_{j>0} \xi_{-j}\lambda^{-j} + \bar{\xi}_{-j}\lambda^j,$$
$$\pi_-(\xi) = \pi_{\mathfrak{b}}(\xi_0) + \sum_{j>0} (\xi_j - \bar{\xi}_{-j})\lambda^j.$$

Let (,) be the Killing form on $\mathcal{G}_2^{\mathbb{C}}$. Then

$$\langle \xi, \eta \rangle = \sum_{i+j=0} (\xi_i, \eta_j)$$

is an ad-invariant bilinear form on $\mathcal{L}(\mathcal{G})$. So

$$\mathcal{L}_+(\mathcal{G})^\perp = \{\xi = \sum_j \xi_j\lambda^j \mid \xi_{-j} = -\bar{\xi}_j\}.$$

Let $M = \{\xi = \xi_0 + \xi_{-1}\lambda^{-1} - \bar{\xi}_{-1}\lambda \mid \xi_0 \in \mathfrak{h}_0 \cap (i\mathcal{G}_2), \xi_{-1} \in \mathfrak{h}_{-1}\}$. Note that

$$\pi_+(\xi_0 + \xi_{-1}\lambda^{-1} - \bar{\xi}_{-1}\lambda) = \xi_{-1}\lambda + \bar{\xi}_{-1}\lambda.$$

It is easy to check that $[\xi, \pi_+(\xi)] \in M$ if $\xi \in M$, so M is invariant under the flow $\xi_t = [\xi, \pi_+(\xi)]$. So we can use the Iwasawa loop group factorization to construct solution of (6.2) as described in the AKS theory and get

Theorem 6.3. *Let $A = 2h_0 + h_{-1} + \bar{h}_{-1}$ be a constant with $h_0 \in \mathfrak{h}_0 \cap \mathfrak{g}_2$ and $h_{-1} \in \mathfrak{h}_{-1}$. Then the solution of* (5.3) *with initial value A can be obtained as follows:*

(1) *Set $\xi_0(\lambda) = 2ih_0 + ih_{-1}\lambda^{-1} + i\bar{h}_{-1}\lambda$, and construct $g(t, \lambda)$ such that*

$$\begin{cases} g^{-1}g_t = \xi_0(\lambda), \\ g(0, \lambda) = \mathrm{I}, \end{cases}$$

i.e., $g(t, \cdot)$ is the one-parameter subgroup of ξ_0 in $L^\sigma(\mathrm{G}_2^{\mathbb{C}})$.

(2) *Factor $g(t, \lambda) = g_+(t, \lambda)g_-(t, \lambda)$ such that $g_\pm(t, \cdot) \in L_\pm^\sigma(\mathrm{G}_2^{\mathbb{C}})$.*

(3) *Set $\xi(t, \lambda) = g_+(t, \lambda)^{-1}\xi_0(\lambda)g_+(t, \lambda)$. Then*

$$\xi(t, \lambda) = v_0(t) + v_{-1}(t)\lambda^{-1} + \bar{v}_{-1}(t)\lambda$$

for some $v_0(t) \in \mathfrak{h}_0 \cap (i\mathfrak{g}_2)$ and $v_{-1}(t) \in \mathfrak{h}_{-1}$.

(4) *Set* $u_0 = -iv_0$, $u_{-1} = -iv_{-1}$, *and* $k(t) = g_+(t,1)$. *Then* $k(t) \in G_2$ *and* u_0, u_{-1}, k *satisfy* (5.3) *and* (5.4).

Moreover, $f(s,t) = e^{As}k_1(t)$ *is almost complex in* S^6, *where* $k_1(t)$ *is the first column of* $k(t)$.

References

[1] Adler, M., van Moerbeke, P., *Completely integrable systems, Euclidean Lie algebras and curves*, Adv. Math., **38** (1980), 267-317

[2] Adler, M., van Moerbeke, P., and Vanhaecke, P., *Algebraic integrability, Painlevé Geometry, and Lie algebras*, EMG, **47** (2004), Springer

[3] Bolton, J., Pedit, F., and Woodward, L. M.,*Minimal surfaces and the affine field model*, J. Reine Angew. Math., **459** (1995), 119-150.

[4] Bolton, J., Vrancken, L., and Woodward, L. M., *On almost complex curves in the nearly Kähler 6-sphere*, Quart. J. Math. Oxford Ser. (2), **45** (1994), 407-427.

[5] Bryant, R., *Submanifolds and special structures on the octonions*, J. Differential Geom., **17** (1982), 185-232.

[6] Burstall, F.E., Pedit, F., *Harmonic maps via Adler-Kostant-Symes Theory*, Harmonic maps and Integrable Systems, Vieweg (1994), 221–272

[7] Calabi, E., *Construction and properties of some* 6*-dimensional almost complex manifolds*, Trans. Amer. Math. Soc., **87** (1958), 407–438.

[8] Carberry, E., and McIntosh, I., *Minimal Lagrangian 2-tori in* $\mathbb{C}\mathrm{P}^2$ *come in real families of every dimension*, J. London Math. Soc., **69** (2004) 531-544,

[9] Ejiri, N., *A generalization of minimal cones*, Trans. Amer. Math. Soc., **276** (1983), 347-360.

[10] Guest, M., *Harmonic maps, loop groups, and integrable systems*, Cambridge University Press, (1997)

[11] Harvey, R., and Lawson, H. B., *Calibrated geometries*, Acta Math., **148** (1982), 47–157.

[12] Haskins, M., *Special Lagrangian cones*, Amer. J. Math., **126** (2004) 845-871

[13] Hashimoto, H., Taniguchi, T., and Udagawa, S., *Constructions of almost complex 2-tori of type III in the nearly Kähler 6 sphere*, Differential Geom. Appl., **21** (2004) 127-145

[14] Kollross, A., *Notes on* G_2.

[15] Ma, H., and Ma, Y., *Totally real minimal tori in* $\mathbb{C}\mathrm{P}^2$, Math. Z., **249** (2005) 241-267

[16] McIntosh, I., *Special Lagrangian cones in* $\mathbb{C}^3$ *and primitive harmonic maps*, J. London Math. Soc., **67** (2003) 769-789

[17] Pressley, A., Segal, G. B., *Loop Groups*, Oxford Science Publ., Clarendon Press, Oxford, (1986)

[18] Terng, C.L., *Geometries and symmetries of soliton equations and integrable elliptic systems*, to appear in Surveys on Geometry and Integrable Systems, Advanced Studies in Pure Mathematics, Mathematical Society of Japan, math.DG/0212372

Department of Mathematics, University of California at Irvine, Irvine, CA 92697-3875
E-mail address: skong@math.uci.edu

Department of Mathematics, University of California at Irvine, Irvine, CA 92697-3875
E-mail address: cterng@math.uci.edu

Department of Mathematics, University of Texas at Austin, Austin, TX 78712-0257
E-mail address: ewang@math.utexas.edu